

From Fock's Transformation to de Sitter Space

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Abstract

As in Deformed Special Relativity, we showed recently that the Fock coordinate transformation can be derived from a new deformed Poisson brackets. This approach allowed us to establish the corresponding momentum transformation which keeps invariant the four dimensional contraction $p_\mu x^\mu$. From the resulting deformed algebra, we construct in this paper the corresponding first Casimir. After first quantization, we show by using the Klein-Gordon equation that the spacetime of the Fock transformation is the de Sitter one. As we will see, the invariant length representing the universe radius in the spacetime of Fock's transformation is exactly the radius of the embedded hypersurface representing the de Sitter spacetime.

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1 Introduction

Special relativity was modified in such a way to keep invariant, in addition to the speed of light, a minimal length [1, 2, 3, 4, 5, 6] in the order of the Planck length. The resulting theory is then called Deformed Special Relativity (DSR). The corresponding coordinate transformation is not linear and it is proved later that it can be constructed by deforming the Poisson brackets [7]. By following the same way, we showed recently [8] that also the Fock coordinate transformation [9],

$$t' = \frac{\gamma(t - ux/c^2)}{\alpha_R}, \quad x' = \frac{\gamma(x - ut)}{\alpha_R}, \quad y' = \frac{y}{\alpha_R}, \quad z' = \frac{z}{\alpha_R}, \quad (1)$$

where

$$\alpha_R = 1 + \frac{1}{R} \left[(\gamma - 1)ct - \gamma \frac{ux}{c} \right], \quad (2)$$

R being the universe radius and $\gamma = (1 - u^2/c^2)^{-1/2}$, can be derived from a new appropriate deformation of the Poisson brackets

$$\{x^\mu, x^\nu\} = 0, \quad (3)$$

$$\{x^\mu, p^\nu\} = -\eta^{\mu\nu} + \frac{1}{R} \eta^{0\nu} x^\mu, \quad (4)$$

$$\{p^\mu, p^\nu\} = -\frac{1}{R} (p^\mu \eta^{0\nu} - p^\nu \eta^{\mu 0}), \quad (5)$$

where $\eta^{\mu\nu} = (+1, -1, -1, -1)$. Here there are c and R which are invariant. We stress that c is a constant with a dimension of a velocity and it represents the light speed only in the limit $R \rightarrow \infty$ [8]. The above brackets allowed us to establish the corresponding momentum transformation

$$E' = \alpha_R \gamma (E - up_x), \quad p'_x = \alpha_R \gamma (p_x - uE/c^2), \quad p'_y = \alpha_R p_y, \quad p'_z = \alpha_R p_z, \quad (6)$$

which keeps invariant the four dimensional contraction $p_\mu x^\mu$. Contrary to earlier versions [6, 10], transformation (6) allows a coherent description of plane waves. We observe that in the limit $R \rightarrow \infty$, (1) and (6) reduce to the Lorentz transformations for the coordinates and the energy-momentum vector. We will call relations (3), (4) and (5) "R-Minkowski phase space algebra".

Furthermore, we showed that the following expressions

$$I_x \equiv \left(1 - \frac{ct}{R}\right)^{-2} \eta_{\mu\nu} x^\mu x^\nu \quad (7)$$

and

$$I_p \equiv \left(1 - \frac{ct}{R}\right)^2 \eta_{\mu\nu} p^\mu p^\nu \quad (8)$$

are invariant under transformations (1) and (6). We also observed that in the limit $R \rightarrow \infty$, I_x and I_p reduce to the well-known invariants of special relativity [8].

Our goal is to establish a link between the spacetime of Fock's coordinate transformation and the one of de Sitter. First, we remark that I_p is not a Casimir. In fact, by using (4) and (5) we can check that it does not commute

with the R -Lorentz group generators p^0 and p^i . In order to construct the first Casimir of the R -algebra, it is necessary to complete relations (3), (4) and (5) with others between pure rotation,

$$M_i = \frac{1}{2}\epsilon_{ijk}J_{jk}, \quad (9)$$

and boost,

$$\tilde{N}_i = J_{0i}, \quad (10)$$

generators. In (9) and (10), $J_{\mu\nu} \equiv x_\mu p_\nu - x_\nu p_\mu$ represents the angular momentum, $(\mu, \nu, \dots = 0, 1, 2, 3, i, j, \dots = 1, 2, 3)$ and ϵ_{ijk} is the Levi-Civita antisymmetric tensor ($\epsilon_{123} = 1$). With the help of the above generators, it is not yet possible to construct the first Casimir. In order to go further, we will follow the method presented by Magpantay [11, 12], where he developed the physics of the dual kappa Poincaré algebra. Then, we will modify expression (10) for boost generator \tilde{N}_i in such a way to make a Casimir construction possible. That's what we will do in the next section. After first quantization in section 3, we will show that the spacetime of the Fock transformation is identical to the de Sitter one. Section 4 is devoted to conclusion.

2 The Casimir construction

We define new boost generators

$$N_i \equiv \tilde{N}_i - \frac{1}{2R}\eta_{\mu\nu}x^\mu x^\nu p_i = x_0 p_i - x_i p_0 - \frac{1}{2R}\eta_{\mu\nu}x^\mu x^\nu p_i \quad (11)$$

which reproduce the usual ones in the limit $R \rightarrow \infty$. We observe that the infinitesimal transformation of any function $O(x^\mu, p^\nu)$, defined as in usual Lorentz transformation

$$\delta O = \left\{ -\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu}, O \right\}, \quad (12)$$

is not affected by the additional term of N_i with respect to \tilde{N}_i because of the antisymmetric feature of the infinitesimal parameters $\omega_{\mu\nu}$. Deformation (11) is reminiscent of the one proposed in [12]. The R -algebra (3), (4) and (5) must be completed by the following brackets

$$\{N_i, p_0\} = -p_i + \frac{N_i}{R}, \quad (13)$$

$$\{N_i, p_j\} = \eta_{ij}p_0 - \frac{1}{R}\epsilon_{ijk}M_k, \quad (14)$$

$$\{M_i, p_0\} = 0, \quad (15)$$

$$\{M_i, p_j\} = \epsilon_{ijk}p_k, \quad (16)$$

$$\{M_i, M_j\} = \epsilon_{ijk}M_k, \quad (17)$$

$$\{M_i, N_j\} = \epsilon_{ijk}N_k, \quad (18)$$

$$\{N_i, N_j\} = -\epsilon_{ijk}M_k, \quad (19)$$

which can be checked after a tedious calculation. We point out that relations (5) and (13)-(19) constitute a particular case of the Bacry Lévy-Leblond algebras presented in [13]. As the Casimirs are scalars and observing that

$$M_i p_i = \epsilon_{ijk}x_j p_k p_i = 0, \quad (20)$$

the first Casimir C can be constructed by combining $p_\mu p^\mu$, $M^i M^i$, $N^i N^i$, $M^i N^i$ and $N^i p^i$

$$C = p_\mu p^\mu + \alpha M^i M^i + \beta N^i N^i + \gamma N^i p^i + \lambda M^i N^i, \quad (21)$$

where the coefficients α , β , γ and λ must be determined by imposing to C to commute with all the generators.

At this step, x^μ and p^μ will be substituted by the corresponding operators $x^\mu \mapsto \hat{x}^\mu$ and $p^\mu \mapsto \hat{p}^\mu$. Expression (11) of the generator N_i is then ill defined because of the commutation relations of \hat{x}^i with \hat{p}^0 and \hat{p}^i . The symmetrization operation compel us to rewrite it as

$$N_i = \hat{x}_0 \hat{p}_i - \frac{1}{2} (\hat{x}_i \hat{p}_0 + \hat{p}_0 \hat{x}_i) - \frac{1}{2R} \hat{x}_0^2 \hat{p}_i + \frac{1}{4R} (\hat{x}^j \hat{x}^j \hat{p}_i + \hat{p}_i \hat{x}^j \hat{x}^j). \quad (22)$$

The Poisson brackets will be replaced by commutators in the following rule

$$\{ \} \mapsto \frac{1}{i\hbar} [\]. \quad (23)$$

The R -Minkowski phase space algebra now reads

$$[\hat{x}^\mu, \hat{x}^\nu] = 0, \quad (24)$$

$$[\hat{x}^0, \hat{p}^0] = -i\hbar \left(1 - \frac{\hat{x}^0}{R} \right), \quad (25)$$

$$[\hat{x}^0, \hat{p}^i] = 0, \quad (26)$$

$$[\hat{x}^i, \hat{p}^0] = i\hbar \frac{\hat{x}^i}{R}, \quad (27)$$

$$[\hat{x}^i, \hat{p}^j] = -i\hbar \eta^{ij}, \quad (28)$$

$$[\hat{p}^i, \hat{p}^j] = 0, \quad (29)$$

$$[\hat{p}^i, \hat{p}^0] = -i\hbar \frac{\hat{p}^i}{R}, \quad (30)$$

and the resulting R -Poincaré algebra takes the form

$$[N_i, \hat{p}_0] = -i\hbar \hat{p}_i + i\hbar \frac{N_i}{R}, \quad (31)$$

$$[N_i, \hat{p}_j] = i\hbar \eta_{ij} \hat{p}_0 - \frac{i\hbar}{R} \epsilon_{ijk} M_k, \quad (32)$$

$$[M_i, \hat{p}_0] = 0, \quad (33)$$

$$[M_i, \hat{p}_j] = i\hbar \epsilon_{ijk} \hat{p}_k, \quad (34)$$

$$[M_i, M_j] = i\hbar \epsilon_{ijk} M_k, \quad (35)$$

$$[M_i, N_j] = i\hbar \epsilon_{ijk} N_k, \quad (36)$$

$$[N_i, N_j] = -i\hbar \epsilon_{ijk} M_k, \quad (37)$$

Relations (31)-(37) obtained in the context of the Fock transformation are identical to that presented in [11, 12] within the framework of the dual kappa Poincaré algebra. Because of the above relations of commutations, expression (21) of the Casimir C must be symmetrized. Therefore, we write

$$C = \hat{p}_\mu \hat{p}^\mu + \alpha M^i M^i + \beta N^i N^i + \frac{\gamma}{2} (N^i \hat{p}^i + \hat{p}^i N^i) + \frac{\lambda}{2} (M^i N^i + N^i M^i). \quad (38)$$

Imposing

$$[C, \hat{p}_0] = 0 \quad (39)$$

leads to take $\beta = 0$, $\lambda = 0$ and $\gamma = 2/R$. Condition

$$[C, \hat{p}^i] = 0 \quad (40)$$

gives $\alpha = -1/R^2$ and expression (38) turns out to be

$$C = \hat{p}_0^2 - \hat{p}^i \hat{p}^i - \frac{1}{R^2} M^i M^i + \frac{1}{R} (N^i \hat{p}^i + \hat{p}^i N^i). \quad (41)$$

One can check that

$$[C, M^i] = 0, \quad [C, N^i] = 0, \quad (42)$$

meaning that expression (41) of C commute with all the generators of the R -Lorentz group. Of course, in the limit $R \rightarrow \infty$, expression (41) reduces to the first Poincaré Casimir. Contrary to DSR theory, we note that the Casimir depends on boost and rotation generators. Identifying R to $1/\kappa$, result (41) is identical to the one obtained by Magpantay [12] in the context of the dual kappa Poincaré algebra.

3 Towards the de Sitter spacetime

Obviously, the usual representation

$$\hat{x}^\mu = x^\mu, \quad (43)$$

$$\hat{p}^\mu = i\hbar \frac{\partial}{\partial x_\mu}, \quad (44)$$

does not work. We can check that the complete algebra (24)-(37) is satisfied if we adopt the following representation:

$$\hat{x}^\mu = x^\mu, \quad (45)$$

$$\hat{p}^0 = i\hbar \left(\frac{\partial}{\partial x^0} - \frac{x^\mu}{R} \frac{\partial}{\partial x^\mu} \right), \quad (46)$$

$$\hat{p}^i = -i\hbar \frac{\partial}{\partial x^i}. \quad (47)$$

We note that expression (46) differs from the one proposed by Magpantay [12] by an additional term. From (46) and (47), we obtain

$$\begin{aligned} \hat{p}^0 \hat{p}^0 = -\hbar^2 \left[\frac{1}{R} \left(-1 + \frac{x^0}{R} \right) \frac{\partial}{\partial x^0} + \frac{x^i}{R^2} \frac{\partial}{\partial x^i} + \left(1 - \frac{x^0}{R} \right)^2 \frac{\partial^2}{\partial (x^0)^2} \right. \\ \left. - 2 \frac{x^i}{R} \left(1 - \frac{x^0}{R} \right) \frac{\partial}{\partial x^0} \frac{\partial}{\partial x^i} + \frac{x^i x^j}{R^2} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \right], \end{aligned} \quad (48)$$

and

$$\hat{p}^i \hat{p}^i = -\hbar^2 \Delta. \quad (49)$$

Since

$$M^i = \frac{1}{2} \epsilon^{ijk} J^{jk} = \epsilon^{ijk} \hat{x}^j \hat{p}^k, \quad (50)$$

and taking into account relation

$$\epsilon^{ijk}\epsilon^{ils} = \delta^{jl}\delta^{ks} - \delta^{js}\delta^{kl}, \quad (51)$$

we obtain with the use of (28)

$$M^i M^i = -\hbar^2 \left[x^i x^i \Delta - x^i x^j \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} - 2x^i \frac{\partial}{\partial x^i} \right]. \quad (52)$$

Relation (32) allows us to write

$$\hat{p}^i N^i = N^i \hat{p}^i + 3i\hbar \hat{p}^0. \quad (53)$$

Using (22), we get to

$$\begin{aligned} N^i \hat{p}^i + \hat{p}^i N^i &= 2N^i \hat{p}^i + 3i\hbar \hat{p}^0 \\ &= -\hbar^2 \left\{ 2 \left[x^0 - \frac{1}{2R} \left((x^0)^2 - x^i x^i \right) \right] \Delta \right. \\ &\quad \left. + 2x^i \left[\left(1 - \frac{x^0}{R} \right) \frac{\partial^2}{\partial x^0 \partial x^i} - \frac{x^j}{R} \frac{\partial^2}{\partial x^j \partial x^i} \right] \right. \\ &\quad \left. + 3 \left[\left(1 - \frac{x^0}{R} \right) \frac{\partial}{\partial x^0} - \frac{x^i}{R} \frac{\partial}{\partial x^i} \right] \right\}. \quad (54) \end{aligned}$$

Substituting (48), (49), (52) and (54) in (41), expression of the first Casimir turns out to be

$$C = -\hbar^2 \left(1 - \frac{x^0}{R} \right)^2 \left(\frac{\partial^2}{(\partial x^0)^2} - \Delta \right) - 2\frac{\hbar^2}{R} \left(1 - \frac{x^0}{R} \right) \frac{\partial}{\partial x^0}. \quad (55)$$

It follows that the Klein-Gordon equation in R -spacetime, $C\phi = m^2 c^2 \phi$, takes the form

$$\left[\hbar^2 \left(1 - \frac{x^0}{R} \right)^2 \left(\frac{\partial^2}{(\partial x^0)^2} - \Delta \right) + 2\frac{\hbar^2}{R} \left(1 - \frac{x^0}{R} \right) \frac{\partial}{\partial x^0} + m^2 c^2 \right] \phi = 0. \quad (56)$$

We can check that this relation represents the Klein-Gordon equation in the de Sitter spacetime, known as

$$\left[\frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} g^{\mu\nu} \partial_\nu \right) + \frac{m^2 c^2}{\hbar^2} \right] \phi = 0, \quad (57)$$

with the following metric

$$ds^2 = \frac{1}{(1 - x^0/R)^2} [(dx^0)^2 - dx^i dx^i]. \quad (58)$$

This result means that the spacetime of the Fock transformation is the same to the de Sitter one in its conformal metric. In fact, if we make the following coordinate transformation

$$x^0 \rightarrow c\tau = x^0 - R, \quad (59)$$

expression (58) takes the form

$$ds^2 = \frac{R^2}{c^2 \tau^2} [c^2 d\tau^2 - dx^i dx^i], \quad (60)$$

which is the conformal metric of the de Sitter spacetime [14, 15]. Furthermore, expression (60) indicates that the invariant length R representing the universe radius in the Fock transformation spacetime is exactly the radius of the embedded hypersurface representing the de Sitter spacetime. Also, we observe that if we compare (58) with the conformal metric presented in [15], we deduce that the Hubble constant is equal to $H = c/R$. This observation strongly reinforces the fact that R is interpreted as the universe radius in the context of the Fock transformation. These results are expected because of the presence of the parameter R in the theory. In fact, it is known that in vacuum the radius R induces the presence of the cosmological constant Λ since this latter is intimately linked to R by $\Lambda = 3/R^2$. But precisely the solutions of Einstein's equations in presence of the cosmological constant are the de Sitter space. This indicates that the symmetry group of the model presented here is a de Sitter group [16] and the R -Minkowski spacetime is indeed a de Sitter one.

We would like to add that other authors have already investigated the de Sitter Special Relativity [17, 18, 19]. Their approach consists in covering the de Sitter space by Beltrami coordinate patches which allow to express easily the law of inertia since geodesics are represented by straight worldlines. Other approaches are also presented in [13, 16].

Also we mention that Magpantay [12] has shown that the spacetime of the dual DSR is identical to the one of the de Sitter in planar coordinates. This result is different from ours since our approach leads straightforwardly to the conformal metric of the de Sitter spacetime. As indicated in [15], it is the conformal metric in form (58) which is compatible with astronomical observations.

4 Conclusion

From the new deformed Poisson brackets which we recently proposed [8] and used to reproduce the Fock coordinate transformation, we constructed in this paper a complete set of commutators of generators and the corresponding first Casimir. Unlike in DSR theory, this Casimir depends on boost and rotation generators. As in [12], its construction was made possible thanks to an appropriate redefinition of the boost generators. After first quantization, we gave a realization of the corresponding deformed algebra and showed by using the Klein-Gordon equation that the spacetime of the Fock transformation is identical to the de Sitter one in its conformal metric, which is compatible with the astronomical observations [15]. As we have seen, the invariant length representing the universe radius in the framework of Fock's transformation is exactly the radius of the embedded hypersurface representing the de Sitter spacetime. The same expression for the first Casimir and similar conclusions, with some nuances, are obtained by Magpantay [12] within the framework of dual DSR. This means that the dual kappa Poincaré algebra deals with the Fock transformation. In the light of the results presented in this work, we strongly suggest that the analogous of the Lorentz transformation in the Minkowski space is the Fock transformation in the de Sitter space.

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